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**DEVELOPMENT OF HIGHER-ORDER MODAL METHODS FOR
TRANSIENT THERMAL AND STRUCTURAL ANALYSIS**

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ABSTRACT

The paper presents and evaluates a force-derivative method which produces higher-order modal solutions to transient problems. These higher-order solutions converge to an accurate response using fewer degrees-of-freedom (eigenmodes) than lower-order methods such as the mode-displacement or mode-acceleration methods. Results are presented for non-proportionally damped structural problems as well as thermal problems modeled by finite elements.

SYMBOLS

C	damping matrix
E	modulus of elasticity
e	spatial error norm (eq. (34))
K	stiffness matrix
\bar{K}	generalized stiffness matrix, conductance matrix
M	mass matrix, moment
m	subset of total number of degrees-of-freedom
\bar{M}	generalized mass matrix, capacitance matrix
n	total number of degrees-of-freedom
Q	force vector
\bar{Q}	generalized force or thermal load vector
q	modal coordinates of the second-order system
t	time
u	displacement
x	coordinate direction
Y	generalized displacement or temperature vector
Z	modal coordinates of the first-order system

Greek

- α_i i th damped eigenvalue (eqs. (3) and (4))
- $[\alpha]$ matrix of damped eigenvalues
- Φ_i i th damped eigenvector
- $[\Phi]$ matrix of damped eigenvectors
- ϕ_i i th normal eigenvector (eq. (21))
- ϵ time-integrated error norm defined in eq. (33)
- Ω matrix of frequencies squared (eq. (23))
- ω_i i th circular natural frequency
- ω_{di} i th circular frequency of the damped free vibration
- Λ matrix of damping coefficients (eq. (23))
- ρ density
- τ dummy variable of integration, temporal integral limit (eq. (33))
- ζ_i i th modal viscous damping factor

Subscripts

- o initial

Superscripts

- a approximate
- (i) ith derivative with respect to time
- \wedge matrix of reduced number of eigenvectors or eigenvalues
or reduced number of modal coordinates

INTRODUCTION

Transient thermal and structural analyses of complicated engineering problems which are modeled as discrete multidegree-of-freedom systems often require the solution of very large systems of equations. Reducing the order of such systems is highly desirable from the standpoint of increased computational efficiency. Some of the many methods for reducing the order of discrete multidegree-of-freedom structural dynamic systems include mass condensation methods (e.g., refs. 1 and 2) and reduced basis methods (e.g., refs. 3-7). The reduced basis methods use either a truncated set of basis vectors (e.g. eigenmodes, Ritz vectors, or Lanczos vectors) or a combination of basis vectors (e.g., eigenmodes and Ritz vectors (ref. 8)). Reduced basis methods have also been applied in solving transient thermal problems (refs. 9-11). Some problems, such as the thermal/structural analysis of space transportation systems or large space structures, require a large number of basis vectors to accurately represent the transient response. In addition, when singularities occur in the loading, convergence of a solution is not guaranteed. For most transient thermal problems, it takes a large number of eigenmodes to accurately model the sharp thermal gradients within the domain (ref. 10).

When a reduced basis method uses the eigenmodes of a system of equations the method is referred to as a modal method. Two of the most widely used modal methods for transient structural analysis are the mode-displacement method (MDM) and the mode-acceleration method (MAM). It was shown in reference 12, that the MAM can be considered a higher-order modal method than the MDM and that the MAM converges to an accurate solution using fewer modes than the MDM (refs. 6 and 12). A method for generating improved or higher-order modal methods was developed in reference 7. This method was generalized to proportionally-damped structures (the damping matrix is a linear function of the stiffness and mass matrices) and evaluated for a variety of loads and damping ratios (ref. 12). This new method successively integrates-by-parts the convolution integral form of the solution and is called the force-derivative method (FDM) because it produces terms which are related to the forcing function and its time derivatives. The FDM was found to be more accurate than either the MDM or MAM. In particular, for problems in which there are a large number of closely-spaced frequencies (e.g., large truss-type space structures or multispan beams) the FDM is very effective in representing the important, but otherwise neglected, higher modes. Recent work indicates that the FDM, which was based on reference 7, has several

variants (refs. 13 and 14) for solving transient structural problems. One such variant is called the dynamic-correction method (DCM) (ref.13) which is useful when a particular solution exists for a given forcing function.

The purpose of the present paper is to extend the work of reference 12 to include non-proportionally damped structural systems and to evaluate the usefulness of the FDM in solving non-proportionally damped structural problems as well as thermal problems. Modal methods evaluated include the MDM, MAM, FDM, and the DCM.

THEORY

First-Order or Damped-Mode Formulation

The equations of motion, in matrix form, of an n degrees of freedom system, together with the initial conditions are given by

$$M\ddot{u} + C\dot{u} + K u = Q(t) \quad (1)$$

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0$$

where M , C , and K are the mass, damping, and stiffness matrices of the system; u and Q are the displacement and load vectors, respectively, and a dot denotes differentiation with respect to time.

Transforming eq. (1) to first-order form results in the following system of equations:

$$\bar{M}\dot{Y} + \bar{K} Y = \bar{Q} \quad Y(0) = Y_0 \quad (2)$$

where

$$Y = \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix}, \quad \bar{K} = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}$$

and

$$\bar{Q} = \begin{Bmatrix} 0 \\ Q \end{Bmatrix} \quad , \quad Y_o = \begin{Bmatrix} \dot{u}_o \\ u_o \end{Bmatrix}$$

Assuming a solution to the homogenous form of eq. (2) as:

$$Y(t) = e^{\alpha_r t} \Phi_r \quad (3)$$

results in an eigenvalue problem

$$\alpha_r \bar{M} \Phi_r + \bar{K} \Phi_r = 0. \quad (4)$$

For distinct eigenvalues (α_r), the eigenvectors (Φ_r) are normalized such that

$$\Phi_r^T \bar{M} \Phi_r = 1.0$$

and then (5)

$$\Phi_r^T \bar{K} \Phi_r = -\alpha_r$$

Equation (5) can also be written in matrix form as

$$[\Phi]^T \bar{M} [\Phi] = [I] \quad \text{and} \quad [\Phi]^T \bar{K} [\Phi] = [\alpha]$$

where $[\Phi]$ is a $2n$ by $2n$ modal matrix with its i th column equal to Φ_r and $[\alpha]$ is a diagonal matrix consisting of the α_r 's.

A solution to eq. (2) is assumed in the form of the following modal summation

$$Y(t) = \sum_{r=1}^{2n} \Phi_r Z_r(t) \quad (6)$$

Substituting eq. (6) into eq. (2) with premultiplication by Φ_r^T results in the following, uncoupled, system of equations:

$$\dot{Z}_r - \alpha_r Z_r = \Phi_r^T \bar{Q} \quad (7)$$

$$Z_r(0) = Z_{r0} = \Phi_r^T \bar{M} Y_0$$

The solution to eq. (7) is

$$Z_r(t) = Z_{r0} e^{\alpha_r t} + \int_0^t e^{\alpha_r(t-\tau)} \Phi_r^T \bar{Q}(\tau) d\tau \quad (8)$$

Hence, the solution of eq. (2) becomes

$$Y(t) = \sum_{r=1}^{2n} \Phi_r \left[Z_{r0} e^{\alpha_r t} + \int_0^t e^{\alpha_r(t-\tau)} F_r(\tau) d\tau \right] \quad (9)$$

where

$$F_r(\tau) = \Phi_r^T \bar{Q}(\tau)$$

If the forcing function has continuous derivatives, the convolution integral of eq. (9) can be integrated by parts to produce higher-order modal methods (ref. 12). For example, if it is integrated by parts once, the following expression results

$$Y(t) = \sum_{r=1}^{2n} \Phi_r \left[Z_{ro} + \frac{F_r(0)}{\alpha_r} \right] e^{\alpha_r t} - \frac{\Phi_r F_r(t)}{\alpha_r} + \frac{\Phi_r}{\alpha_r} \int_0^t e^{\alpha_r(t-\tau)} \dot{F}_r(\tau) d\tau \quad (10)$$

If all the modes are used in the second-to-last term in eq. (10), this term can be written as

$$\sum_{r=1}^{2n} -\Phi_r \left[\frac{1}{\alpha_r} \right] F_r(t) = -[\Phi] \left[\frac{1}{\alpha} \right] [\Phi]^T \bar{Q}(t) = \bar{K}^{-1} \bar{Q}(t) \quad (11)$$

if equation (11) is substituted into equation (10), the resulting expression is analogous to the MAM (ref. 6) and can be written as

$$Y(t) = \sum_{r=1}^{2n} \left\{ \Phi_r \left[Z_{r0} + \frac{F_r(0)}{\alpha_r} \right] e^{\alpha_r t} + \frac{\Phi_r}{\alpha_r} \int_0^t e^{\alpha_r(t-\tau)} \dot{F}_r(\tau) d\tau \right\} + \bar{K}^{-1} \bar{Q}(t) \quad (12)$$

If the forcing function has continuous derivatives up to order N (C^N), the convolution integral of eq. (9) can be integrated-by-parts N -times, resulting in the following expression

$$Y(t) = \sum_{r=1}^{2n} \left\{ \Phi_r \left[Z_{r0} + \sum_{i=1}^N \frac{F_r^{(i-1)}(0)}{\alpha_r^i} \right] e^{\alpha_r t} + \frac{\Phi_r}{\alpha_r^N} \int_0^t e^{\alpha_r(t-\tau)} F_r^{(N)}(\tau) d\tau - \sum_{i=1}^N \frac{\Phi_r F_r^{(i-1)}(t)}{\alpha_r^i} \right\} \quad (13)$$

where the superscript $(i-1)$ denotes the $(i-1)$ th derivative.

If all the modes are used in the last N -terms of eq. (13), they can be represented as functions of \bar{M} and \bar{K} (ref. 12) as follows

$$\begin{aligned}
Y(t) = & \sum_{r=1}^{2n} \left\{ \Phi_r \left[Z_{r0} + \sum_{i=1}^N \frac{F_r^{(i-1)}(0)}{\alpha_r^i} \right] e^{\alpha_r t} + \frac{\Phi_r}{\alpha_r^N} \int_0^t e^{\alpha_r(t-\tau)} F_r^{(N)}(\tau) d\tau \right\} \\
& + \sum_{i=1}^N \left(-\bar{K}^{-1} \bar{M} \right)^{i-1} \bar{K}^{-1} \bar{Q}(t)
\end{aligned} \tag{14}$$

Alternate Damped-Mode Formulation

Equations 8 to 14 are analogous to those presented in reference 12, however, the present expressions solve a first-order system of equations, using the damped modes Φ_r to decouple a non-proportionally damped system (the damping matrix is not a linear function of the mass and stiffness matrices). This means the order of the system of equations is doubled ($2n$). Equation 14 represents a means for developing higher-order modal methods than either the MDM or MAM, as demonstrated in reference 12, and is called the force-derivative method (FDM) because it produces terms which are related to the forcing function and its time derivatives.

Equation 2 can also be considered to represent a heat conduction problem where, for that problem, \bar{M} would represent the capacitance matrix, \bar{K} would represent the conductance matrix, \bar{Q} represents the thermal load vector, and Y is the vector of nodal temperatures.

The MDM uses a subset, m ($m < 2n$), of the eigenmodes to reduce the size of the problem and solves for Z_r using eq. (8) and substitutes these values into a reduced modal summation in eq. (6) to approximate the response, $Y(t)$. The MDM can be classified as a zeroth-order method because it is equivalent to using the FDM (eq. (14)) with $N = 0$. An analogous form of the MAM uses eq. (12), and a reduced modal summation to approximate $Y(t)$ and can be classified as a first-order method ($N = 1$ in eq. (14)). The FDM

uses eq. (14) with $N > 1$. Reference 12 showed that the expressions obtained with four integrations by parts offer improved approximations to the higher, neglected, modes for several structural problems.

The FDM (eq. (14)) can be derived using an approach similar to that used in reference 13 which results in a form which is more suitable for inclusion into existing thermal and structural analysis codes. A numerical approach can be derived, similar to that presented in reference 13, which approximates the forcing function as a piecewise differentiable polynomial and which numerically integrates the reduced system of equations (eq. (8)). For example, assume the forcing function is C^2 differentiable. Equation 7 could be differentiated twice to produce the following equations

$$\ddot{Z}_r - \alpha_r \dot{Z}_r = \Phi_r^T \dot{\ddot{Q}} \quad (15a)$$

$$Z_r^{(3)} - \alpha_r \ddot{Z}_r = \Phi_r^T \ddot{\ddot{Q}} \quad (15b)$$

Re-arranging eq. (7) and substituting for \dot{Z}_r and \ddot{Z}_r from eqs. (15a and 15b) results in

$$\begin{aligned} Z_r(t) = & -\frac{1}{\alpha_r} \Phi_r^T \bar{Q}(t) - \frac{1}{\alpha_r^2} \Phi_r^T \dot{\bar{Q}}(t) - \frac{1}{\alpha_r^3} \Phi_r^T \ddot{\bar{Q}}(t) \\ & + \frac{1}{\alpha_r^3} Z_r^{(3)}(t) \end{aligned} \quad (16)$$

Using eq. (6), the response can be written as

$$Y(t) = \bar{K}^{-1} \bar{Q}(t) - \bar{K}^{-1} \bar{M} \bar{K}^{-1} \dot{\bar{Q}}(t) + \bar{K}^{-1} \bar{M} \bar{K}^{-1} \bar{M} \bar{K}^{-1} \ddot{\bar{Q}}(t) + \sum_{r=1}^{2n} \Phi_r \frac{1}{\alpha_r^3} Z_r^{(3)}(t) \quad (17)$$

The last term can be evaluated using eq. (8) and Leibnitz's rule for differentiation of an integral to produce the following

$$Z_r^{(3)}(t) = \alpha_r^2 \Phi_r^T \bar{Q}(t) + \alpha_r \Phi_r^T \dot{\bar{Q}}(t) + \Phi_r^T \ddot{\bar{Q}}(t) + \alpha_r^3 Z_{r0} e^{\alpha_r t} + \alpha_r^3 \int_0^t e^{\alpha_r(t-\tau)} \Phi_r^T \bar{Q}(\tau) d\tau \quad (18)$$

$Y(t)$ can be approximated using only a subset of the modes for last term in eq. (17) and using eqs. (8) and (18), eq. (17) becomes

$$Y(t) \equiv \left(\bar{K}^{-1} + \hat{\Phi} \hat{\alpha}^{-1} \hat{\Phi}^T \right) \bar{Q}(t) + \left(-\bar{K}^{-1} \bar{M} \bar{K}^{-1} + \hat{\Phi} \hat{\alpha}^{-2} \hat{\Phi}^T \right) \dot{\bar{Q}}(t) + \left(\bar{K}^{-1} \bar{M} \bar{K}^{-1} \bar{M} \bar{K}^{-1} + \hat{\Phi} \hat{\alpha}^{-3} \hat{\Phi}^T \right) \ddot{\bar{Q}}(t) + \hat{\Phi} \hat{Z}(t) \quad (19)$$

where the $\hat{}$ denotes a reduced set of modes ($m < 2n$) and Z can be calculated by numerically integrating eq. (7)

This expression can be expanded, assuming a C^N differentiable forcing function, to

$$Y(t) \cong \sum_{i=1}^N \left[\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \bar{K} \bar{M} \right)^{i-1} \bar{K}^{-1} + \bar{\Phi} \bar{\alpha}^i \bar{\Phi}^T \right] \bar{Q}(t) + \bar{\Phi} \bar{Z}(t) \quad (20)$$

Compared to eq. (14), the alternate formulation of the FDM (eq. (20)) does not require the solution of a convolution-type integral. In addition, the last term of eq. (20) is identical to a mode-displacement solution and as such the form of the FDM as given by eq. (20) is more suited for inclusion into existing computer codes.

Second-Order or Natural-Mode Formulation

Expressions which use the undamped natural modes can be developed in an analogous manner and result in an expression similar to eqs. (19) and (20). This is accomplished by beginning with the undamped modes of the second-order system of equations (eq. (1)). The modes, ϕ_r , are determined by solving the following eigenvalue problem

$$K \phi_r = \omega_r^2 \phi_r \quad (21)$$

The modes are normalized as follows

$$\phi_r^T M \phi_r = 1.0$$

so that

$$\phi_r^T K \phi_r = \omega_r^2$$

Hence, the displacement response can be represented as

$$u(t) = \sum_{r=1}^n \phi_r q_r(t) \quad (22)$$

Using eq. (22), and premultiplying eq. (1) by ϕ_r^T results in

$$\ddot{q} + \Lambda \dot{q} + \Omega^2 q = [\phi]^T Q(t) \quad (23)$$

where

$$\Lambda = [\phi]^T C [\phi]$$

and

$$\Omega^2 = [\phi]^T K [\phi]$$

where $[\phi]$ is the matrix of undamped eigenmodes, Ω^2 is a diagonal matrix whose diagonal terms can be represented as $\Omega_{ii}^2 = \omega_i^2$ and, for proportional damping, Λ is also a diagonal matrix whose diagonal terms can be represented as $\Lambda_{ii} = 2\zeta_i \omega_i$.

If we assume the forcing function is C^2 differentiable, eq. (23) can be differentiated twice and back substituted into eq. (23) to produce the following expression, which is similar to eq. (16)

$$\begin{aligned}
 q(t) = & \Omega^{-2} [\phi]^T Q(t) - \Omega^{-2} \Lambda \Omega^{-2} [\phi]^T \ddot{Q} + \left[\Omega^{-2} \Lambda \Omega^{-2} \Lambda \Omega^{-2} - \Omega^{-2} \Omega^{-2} \right] \phi^T \ddot{Q} \\
 & + \left[\Omega^{-2} \Omega^{-2} \Lambda - \Omega^{-2} \Lambda \Omega^{-2} \Lambda \Omega^{-2} \Lambda + \Omega^{-2} \Lambda \Omega^{-2} \right] q^{(3)}(t) \\
 & + \left[\Omega^{-2} \Omega^{-2} - \Omega^{-2} \Lambda \Omega^{-2} \Lambda \Omega^{-2} \right] q^{(4)}(t)
 \end{aligned} \tag{24}$$

Assuming proportional damping and zero initial conditions, the solution to eq. (23) can be written as

$$q_r(t) = \frac{1}{\omega_{dr}} \int_0^t e^{-\zeta_r \omega_r(t-\tau)} \sin \omega_{dr}(t-\tau) \phi_r^T Q(\tau) d\tau \tag{25}$$

where

$$\omega_{dr} = \sqrt{\omega_r^2 - (\zeta_r \omega_r)^2}$$

If eq. (25) is differentiated four times with respect to t and the expressions for $q^{(3)}$ and $q^{(4)}$ are substituted into eq. (24), the entire expression reduces to the following

$$\begin{aligned}
u(t) \equiv & \left(K^{-1} - \hat{\phi} \hat{\Omega}^{-2} \hat{\phi}^T \right) \bar{Q}(t) - \left(K^{-1} C K^{-1} - \hat{\phi} \hat{\Omega}^{-2} \hat{\Lambda} \hat{\Omega}^{-2} \hat{\phi}^T \right) \dot{\bar{Q}} \\
& - \left[\left(K^{-1} M K^{-1} - K^{-1} C K^{-1} C K^{-1} \right) - \left(\hat{\phi} \hat{\Omega}^{-2} \hat{\Omega}^{-2} \hat{\phi}^T - \hat{\phi} \hat{\Omega}^{-2} \hat{\Lambda} \hat{\Omega}^{-2} \hat{\Lambda} \hat{\Omega}^{-2} \hat{\phi}^T \right) \right] \ddot{\bar{Q}}(t) \\
& + \hat{\phi} \hat{\Lambda} \hat{q}(t)
\end{aligned} \tag{26}$$

Equation (26) agrees with results presented in reference 13 and, as shown in reference 13, is also valid for non-proportionally damped structural systems. Assuming higher-order piecewise differentiable forcing functions, the method above can produce successively higher-order modal methods and, as such, is just another formulation of the FDM. The expression for an N^{th} -order modal method can be expressed as

$$u(t) \equiv \sum_{r=1}^N (B_{1,r-1} - \hat{A}_{1,r-1}) Q^{(r-1)}(t) + \hat{\phi} \hat{\Lambda} q(t) \tag{27}$$

where

$$B_r = \begin{bmatrix} B_{1,r} \\ B_{2,r} \end{bmatrix} = \begin{bmatrix} -K^{-1} C B_{1,r-1} - K^{-1} M B_{2,r-1} \\ B_{1,r-1} \end{bmatrix}, \quad B_0 = \begin{bmatrix} K^{-1} \\ 0 \end{bmatrix}$$

and

$$\hat{A}_r = \hat{\phi} \begin{bmatrix} A_{1,r} \\ A_{2,r} \end{bmatrix} \hat{\phi}^T = \hat{\phi} \begin{bmatrix} -\hat{\Omega}^{-2} \hat{\Lambda} A_{1,r-1} - \hat{\Omega}^{-2} A_{2,r-1} \\ A_{1,r-1} \end{bmatrix} \hat{\phi}^T, \quad A_0 = \hat{\phi} \begin{bmatrix} \hat{\Omega}^{-2} \\ 0 \end{bmatrix} \hat{\phi}^T$$

The Dynamic-Correction Method (DCM)

The dynamic-correction method (DCM), reference 13, assumes a solution to eq. (2) in the form

$$Y(t) = Y_P(t) + Y_C(t) \quad (28)$$

where $Y_P(t)$ is a particular solution of eq. (2) and $Y_C(t)$ is the complimentary solution which represents the effects of initial conditions. In modal form, $Y_P(t)$ and $Y_C(t)$ can be represented as

$$Y_P(t) = [\Phi] Z_P(t)$$

and (29)

$$Y_C(t) = [\Phi] Z_C(t)$$

where $Z_P(t)$ and $Z_C(t)$ are the vectors of particular and complimentary solutions to the modal coordinate equations (eq. (7)).

Using eqs. (6) and (28)

$$Y(t) = [\Phi] Z(t) = [\Phi][Z(t) - Z_P(t)] + Y_P(t) \quad (30)$$

The fundamental principle of the DCM is that if you have an exact particular solution to eqs. (2) and (7), you can approximate the response (eq. (30)) using a reduced set of modes as shown below

$$Y(t) \cong \hat{\Phi} \hat{Z}(t) + [Y_P(t) - \hat{\Phi} \hat{Z}_P(t)] \quad (31)$$

It can also be shown that in the limit as N goes to infinity, two terms in equation (20) can be written as

$$\lim_{N \rightarrow \infty} = \left\{ -\hat{\Phi} \left[\sum_{r=1}^N \hat{\alpha}^{-r} \Phi^T \bar{Q}(t)^{(r-1)} \right] \right\} = \hat{\Phi} \hat{Z}_P(t)$$

and

(32)

$$\lim_{N \rightarrow \infty} = \left\{ \left[\sum_{r=1}^N \left(\begin{pmatrix} -\bar{K}^{-1} & \bar{M} \end{pmatrix}^{r-1} \bar{K}^{-1} \right) \Phi^T \bar{Q}(t)^{(r-1)} \right] \right\} = Y_P(t)$$

Hence, if an infinite number of integrations-by-parts are assumed in the FDM or if the convolution integral vanishes (e.g., for a polynomial forcing function of a lower order than the order of the FDM) the FDM would be equivalent to the DCM of reference 13. Also, if an exact solution to the convolution integrals of eqs. (8), (12), or (14) exists and is used; the response can be calculated without errors caused by the approximation of the forcing function. For the case of a sinusoidal forcing function, equations (32) will converge as shown above provided the frequency of the highest mode used in the approximation is larger than the forcing function frequency.

RESULTS AND DISCUSSION

Structural Example: Two-Degree-Of-Freedom Problem

A simple, two-degree-of-freedom spring-mass problem (fig. 1) with a sinusoidal forcing function was analyzed to compare the accuracy of the MDM, MAM, FDM, and DCM. This problem was also investigated in reference 13 and included in that reference are the particular solutions for polynomial as well as sinusoidal forcing functions. As shown in figure 1, the sinusoidal forcing function, $\sin(\omega_f t)$, is applied to the second mass. The natural frequencies are $\omega_1 = 19.54$ rad/s and $\omega_2 = 51.17$ rad/s. The system is proportionally damped (Λ (eq. (23)) is diagonal) if $\alpha = K_1/K_2$ (ref. 15). For the stiffnesses chosen, this corresponds to a value of $\alpha = 1.0$. The accuracy of each method is assessed by a time-integrated error norm, which is defined as

$$\epsilon_i (\%) = \frac{\int_0^{\tau} |u_i(t) - u_i^a(t)| dt}{\int_0^{\tau} |u_i(t)| dt} \times 100 \quad (33)$$

where $u_i(t)$ is the calculated response using all the modes and $u_i^a(t)$ is the approximate response using a subset of the modes. Results were calculated using both the real and damped modes (eqs. (20) and (27)) respectively. For this problem, the FDM used was of order four ($N=4$ in eqs. (20) and (27)). The time, τ , selected for integrating the error was chosen to be $\tau = 16\pi/\omega_f$.

Results of the error as a function of the forcing frequency for the undamped modal solution using one real mode for the proportionally-damped case ($\alpha = 1.0$) is shown in figure 2. In general, the forcing function

frequency must be lower than the highest natural frequency used in the approximate modal response for accurate results. As shown in figure 2, the accuracy increases as the order of the modal method increases. The results for the lower range of frequencies are shown more clearly in figure 3, which is an expanded error scale of figure 2, that the FDM ($N=4$) and DCM are similar and more accurate than the lower order methods such as the MDM ($N=0$) and the MAM ($N=1$) for $\omega_f < 20$ rad/s. For $\omega_f > 20$ rad/s, the DCM remains slightly more accurate than the FDM, however, as the forcing frequency approaches the second natural frequency ($\omega_f = 51.17$ rad/s) all methods produce inaccurate results. It should be noted that results using one real mode or two damped modes are identical for the proportionally-damped case.

Results for the non-proportionally-damped case ($\alpha = 20$) using two damped modes, are shown in figure 4. Results are similar to the proportionally-damped case with the exception that the DCM exhibits surprisingly good results at forcing function frequencies close to the second natural frequency. This result is unexplained at present and is believed to be fortuitous and, hence, it is recommended that all modal methods should include modes whose frequencies bound the frequency of the forcing function. A comparison of the damped-mode solution (using two damped modes (eq. (20))) and the undamped solution (using one real mode (eq. (27))) is shown in figure 5 for the non-proportionally-damped case ($\alpha = 20$). As shown in figure 5, the damped-mode solution using two damped modes (dashed lines) produces more accurate results than those using only one real mode (solid lines). The damped-mode solution for the FDM and DCM are nearly equivalent and result in the smallest error for frequencies as large as 30 rad/s. Hence, it may be beneficial, in some cases, to use the damped modes to obtain a more accurate solution.

The FDM produced similar results to the DCM for forcing frequencies below the first natural frequency. A comparison of the modal methods for a forcing frequency $\omega_f = 10$ rad/s is shown in figure 6. Once again, the higher-order modal methods result in more accurate solutions. The large relative errors near $\tau = 0$ are due to the zero initial conditions which cause the denominator of eq. (33) to approach zero at $\tau = 0$. As explained in reference 12, the increase in accuracy with the order of the modal method is due to the addition of terms which are functions of the generalized stiffness and mass matrices and the force vector and its time derivatives. These additional terms approximate the effect of the higher modes which were neglected in the modal summation.

Thermal Example: Rod Heated At One End

Results of references 10 and 11 indicate that Lanczos vectors can be effective reduced basis vectors for solving linear and non-linear thermal problems. Since the accuracy of the Lanczos vectors is comparable to that of the MAM for structural dynamic problems, it was expected that higher-order methods, such as the MAM, FDM and DCM, would be effective in solving complex thermal problems. The problem selected to study is similar to that presented in reference 10 with one exception; the present problem assumes the temperature at the right end of the rod is constrained to zero. The forcing function is a ramp up from zero to a peak value at time $t = 10$ sec and a ramp down to zero at time $t = 20$ sec as shown in figure 7. The error function used to evaluate convergence is a spatial error norm similar to that used in reference 12, namely

$$e = \sqrt{\frac{(T - T^a)^T (T - T^a)}{T^T T}} \quad (34)$$

A total of twenty equally-spaced finite elements were used to model the problem. The temperature distributions in the rod as a function of the number of modes for the MDM, MAM, and FDM are shown in figures 8a to 8c for time $t = 10$ sec. Because the forcing function is linear, the FDM (having an order of 2) is exactly equivalent to the DCM. The FDM converges with 5 modes to the exact solution as compared to 8 modes for the MAM and 18 modes for the MDM. A spatial error norm similar to that shown in eq. (34) is used to compare each method for time, $t = 10$ sec (fig. 9). The effectiveness of using higher-order modal methods for reducing the size and computational effort of thermal problems is illustrated in figure 9. The FDM or DCM require about 28-percent of the number of modes as compared to the MDM and about 63-percent of the number of modes as the MAM for an accurate thermal response.

CONCLUDING REMARKS

The present study extends the development of higher-order modal methods to include non-proportionally damped systems using both first-order (damped modes) and second-order (natural modes) modal-superposition methods. The study compares the accuracy and convergence of the mode-displacement, mode-acceleration, force-derivative, and the dynamic-correction methods (MDM, MAM, FDM, and DCM respectively) in solving proportionally and non-proportionally damped structural dynamic problems as well as transient thermal problems. The higher-order modal methods, such as the FDM or DCM, are very effective in solving structural dynamic problems. The damped mode solutions are found to be effective in solving a non-proportionally damped two-degree-of-freedom problem.

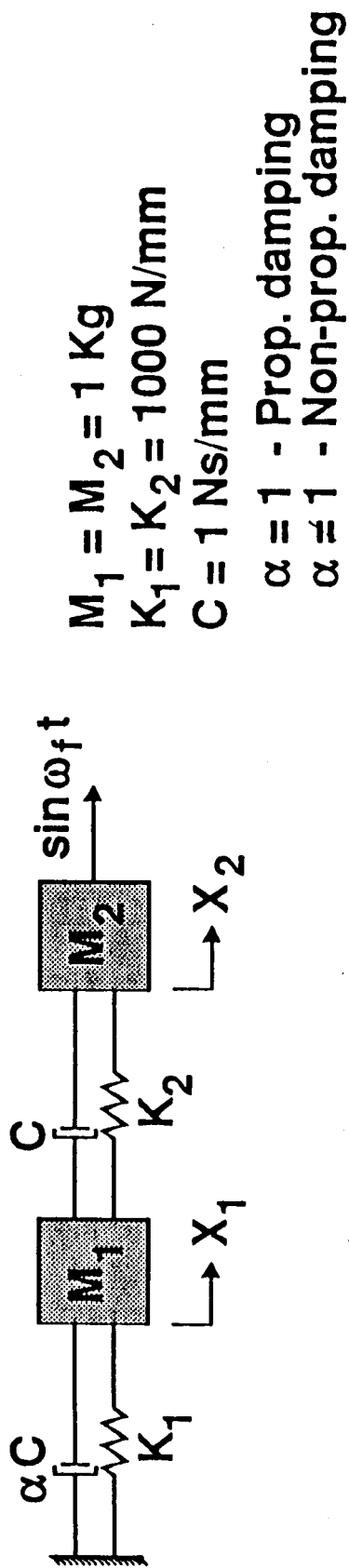
Results of a two-degree-of-freedom spring-mass-damper system indicate that, for the proportionally damped problem, a solution using two damped modes produces identical results as one using only one natural mode. Hence, there is no advantage in using a damped modal solution to solve a proportionally damped problem. However, for the non-proportionally damped problem, the use of two damped modes produces more accurate results than the single natural mode case. The DCM has the lowest percentage error of all the mode-superposition methods over the frequency range of 2 to 50 rad/s. The FDM, having an order of four (four integrations-by-parts ($N=4$)), produced similar results to the DCM up to a forcing frequency of about 35-40 rad/s. For the proportionally damped problem, all the methods were inaccurate near a frequency of 50 rad/s (close to the second natural frequency of the system).

A one-dimensional heat conduction problem (rod heated at one end) was also investigated to evaluate the usefulness of using the FDM to solve thermal problems. The higher-order modal methods such as the FDM were very effective in solving the thermal problem. Until now, modal methods were inefficient in solving thermal problems because the nature of the problem required the inclusion of the higher modes for an accurate solution. The ability of the FDM to approximate the effects of the higher, but neglected, modes resulted in an accurate solution using only five modes out of a total of twenty as compared to the MDM which required 18 modes for an accurate solution.

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Time-integrated error norm

$$\varepsilon_i(\%) = \frac{\int_0^{\tau} |X_i(t) - X_i^a(t)| dt}{\int_0^{\tau} |X_i(t)| dt} \times 100$$

$$\tau = 16\pi/\omega_f$$

X_i^a = approx. response

Figure 1.- Two-degree-of-freedom spring-mass problem.

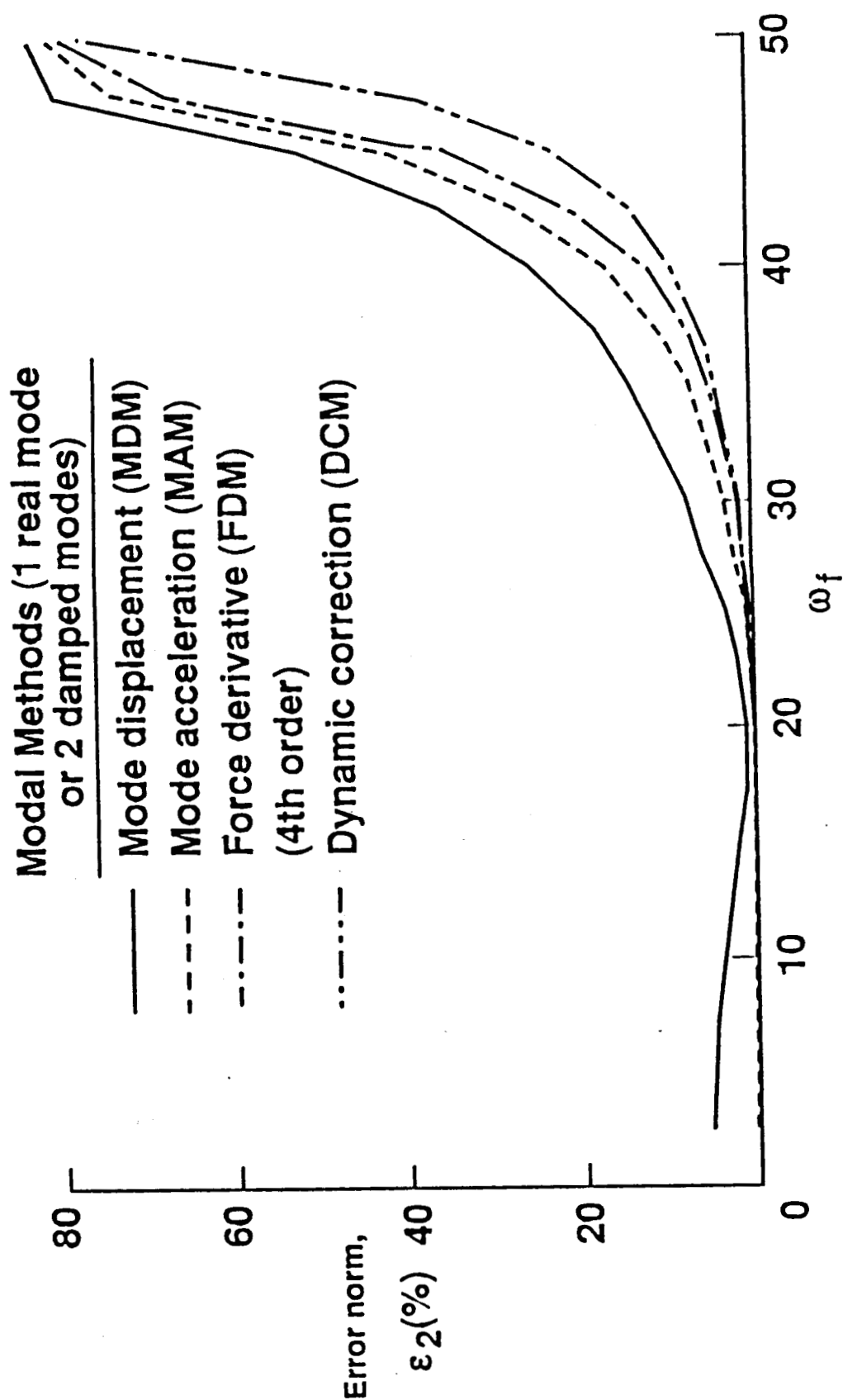


Figure 2.- Comparison of modal methods as a function of forcing frequency for a proportionally-damped two-degree-of-freedom problem ($\alpha = 1$).

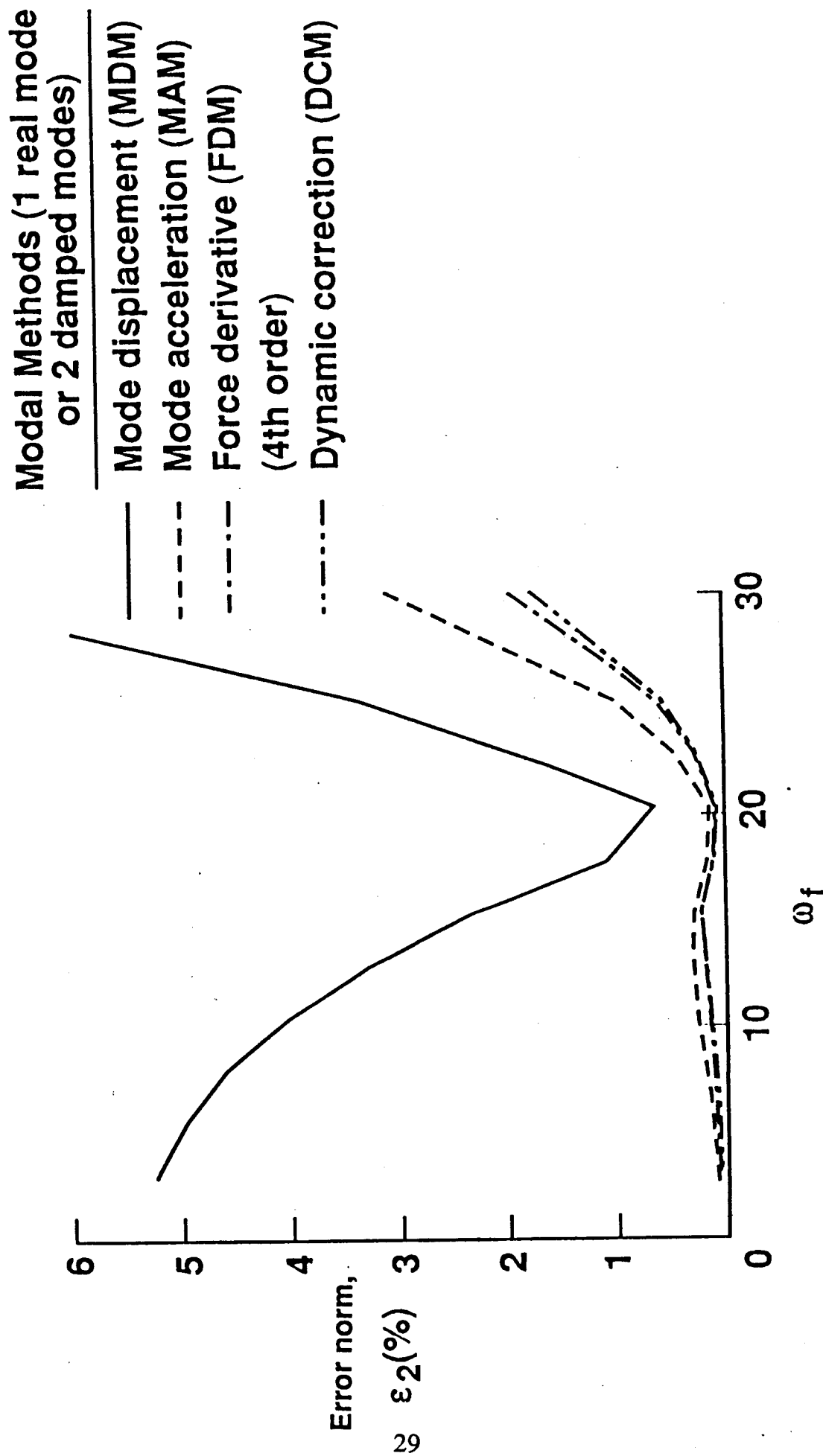


Figure 3.- Comparison of modal methods as a function of forcing frequency for a proportionally-damped two-degree-of-freedom problem ($\alpha = 1$).

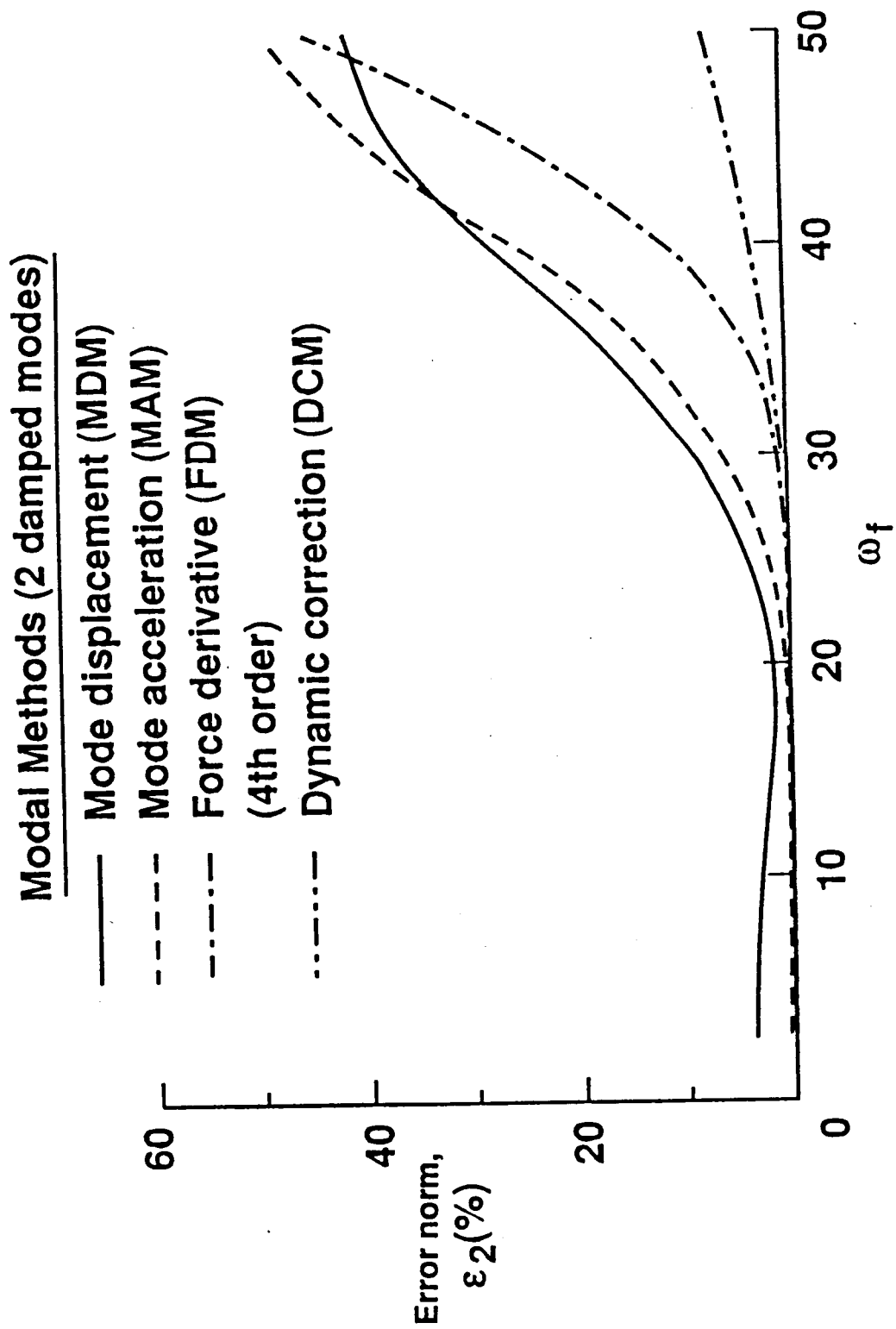


Figure 4.- Comparison of modal methods as a function of forcing frequency for a non-proportionally-damped two-degree-of-freedom problem ($\alpha = 20$).

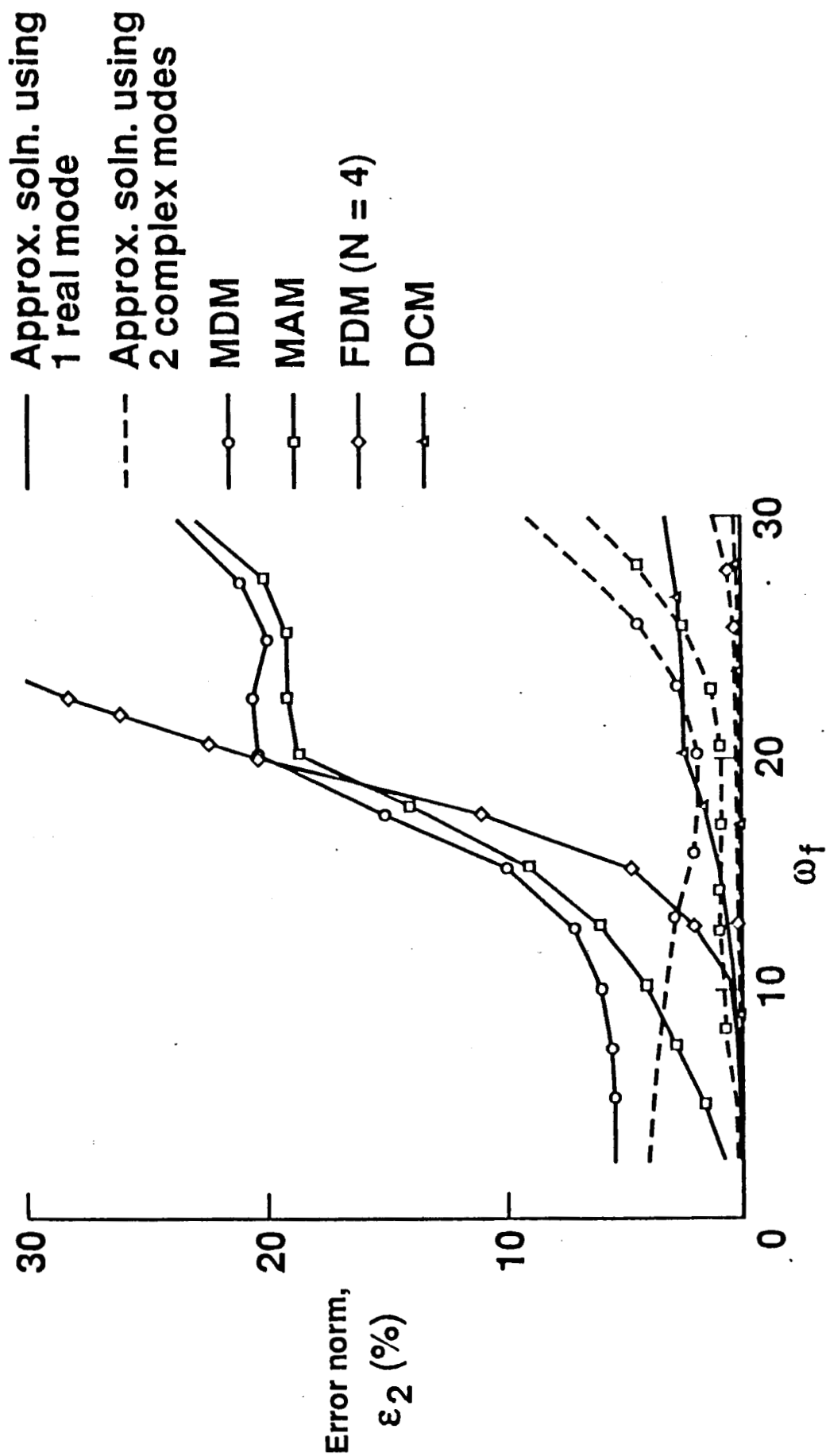


Figure 5.- Comparison of damped and natural mode solutions for a non-proportionally-damped two-degree-of-freedom problem ($\alpha = 20$).

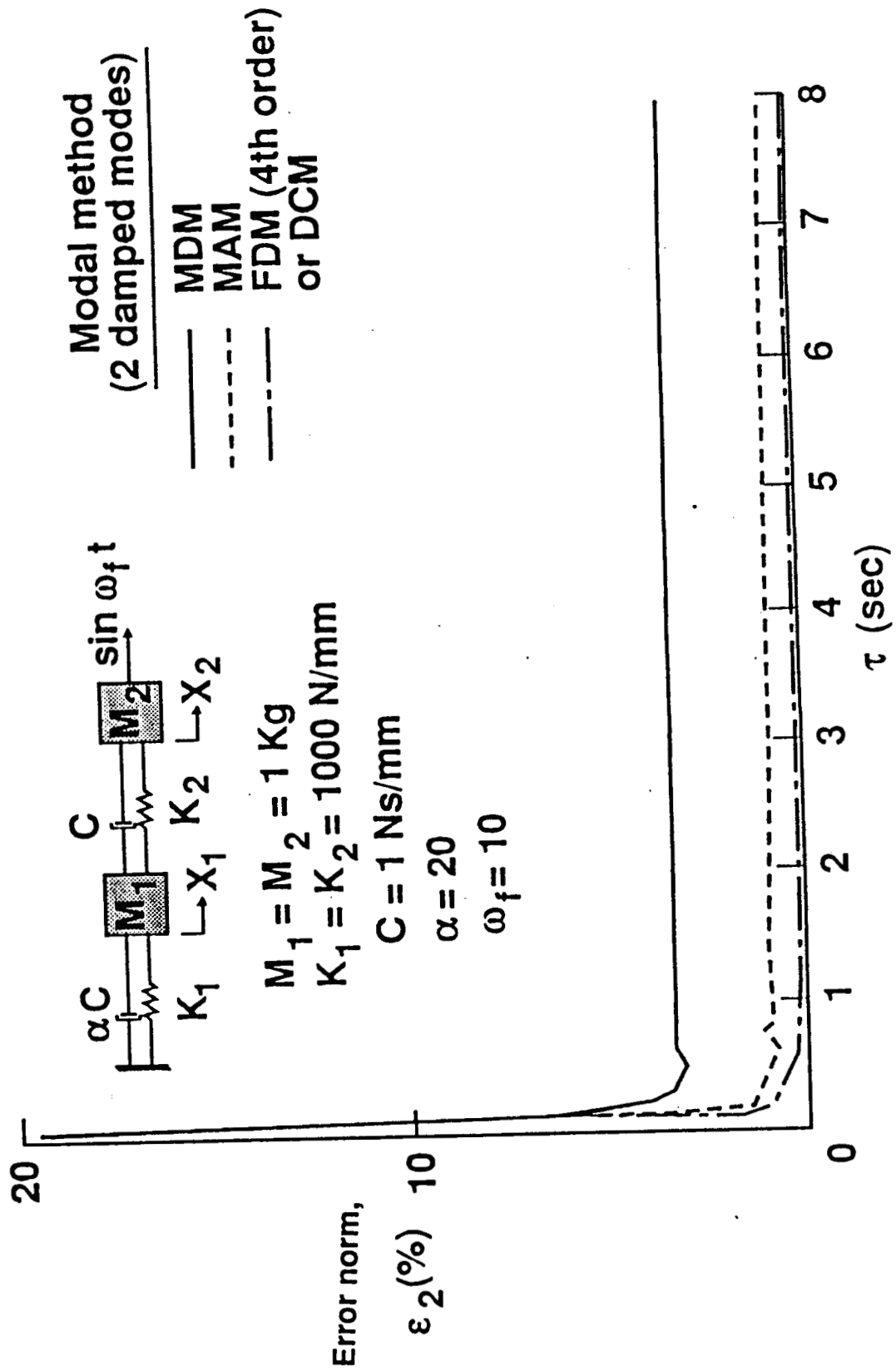
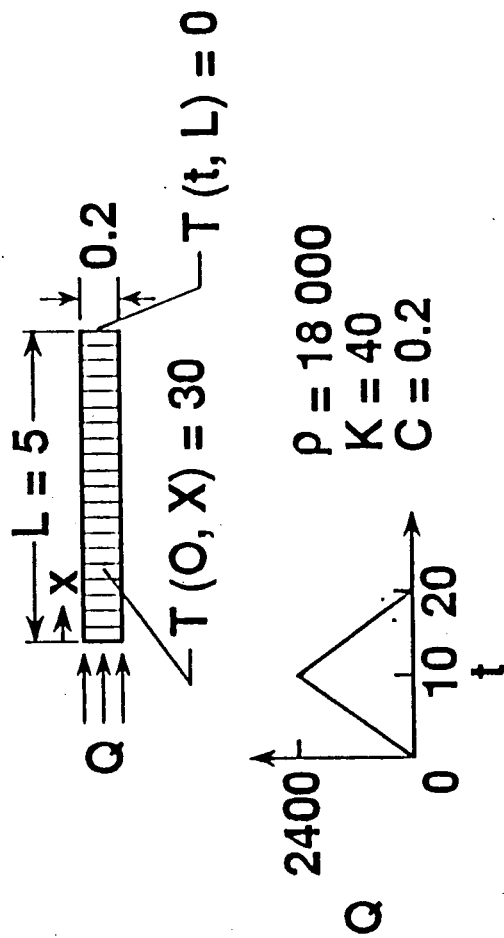


Figure 6.- Comparison of modal methods as a function of time for a non-proportionally-damped problem ($\alpha = 20$) with $\omega_f = 10$.



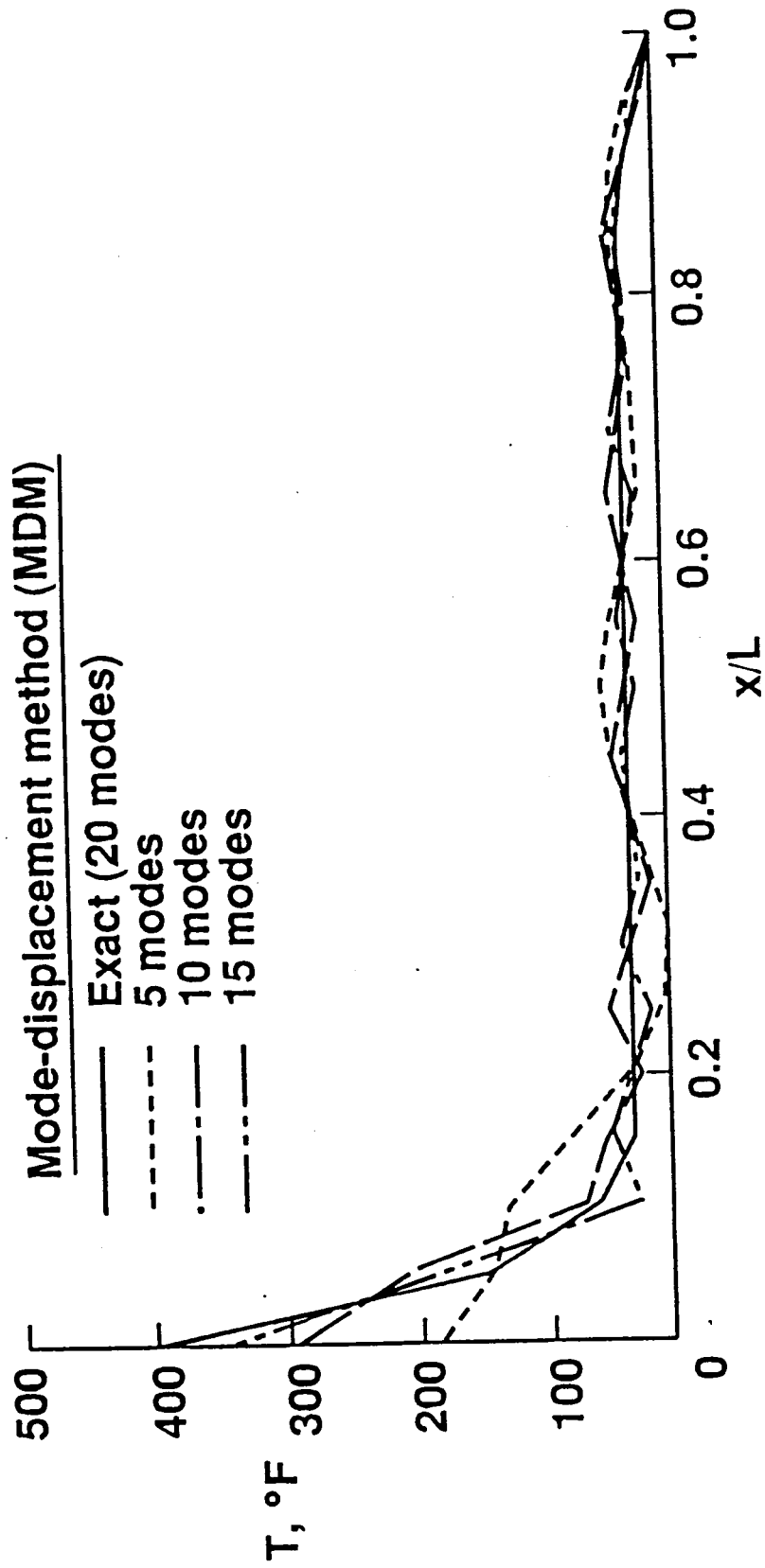
Spatial error norm

$$e = \frac{(T - T^a)(T - T^a)}{T^T T}$$

where T^a = approx. temp. vector

Figure 7.- One-dimensional heat conduction problem: rod heated at one end.

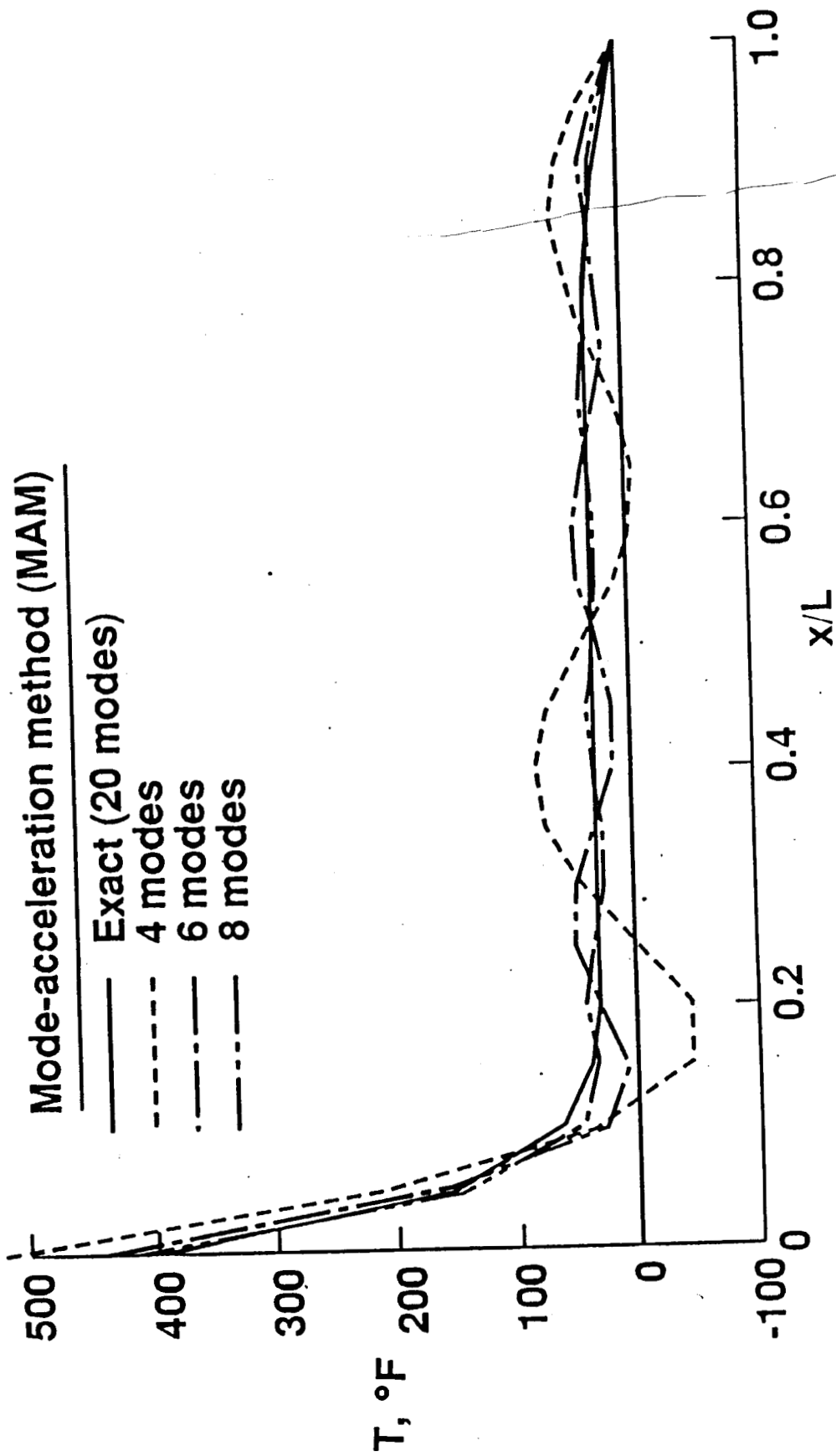
Time = 10 sec



a) Mode-displacement method (MDM)

Figure 8.- Temperature distribution along a rod heated at one end at $t = 10$ sec.

Time = 10 sec

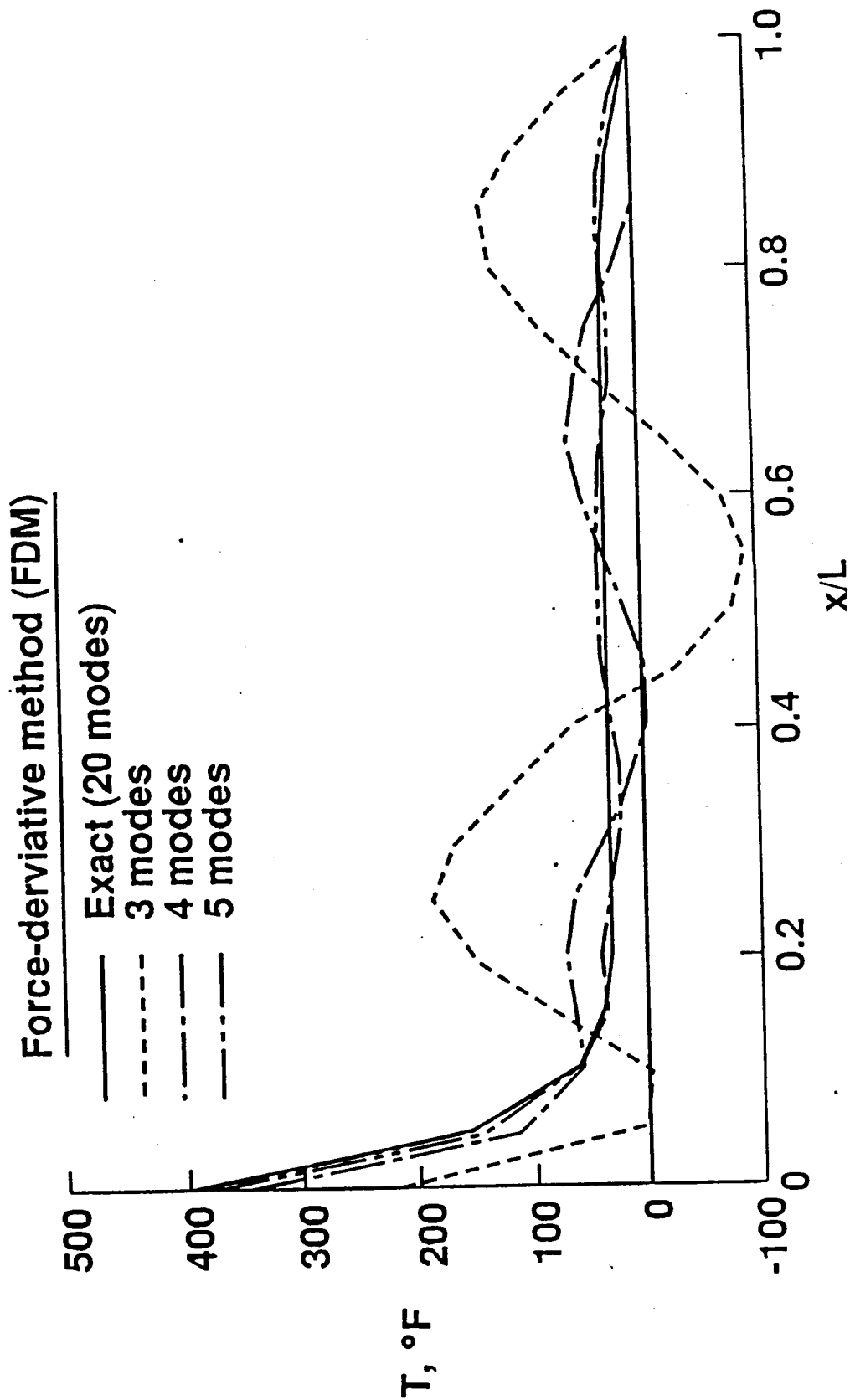


b) Mode-acceleration method (MAM)

Figure 8.- continued.

TEMPERATURE DISTRIBUTION ALONG ROD HEATED AT ONE END

Time = 10 sec



c) Force-derivative method (FDM)

Figure 8.- concluded.

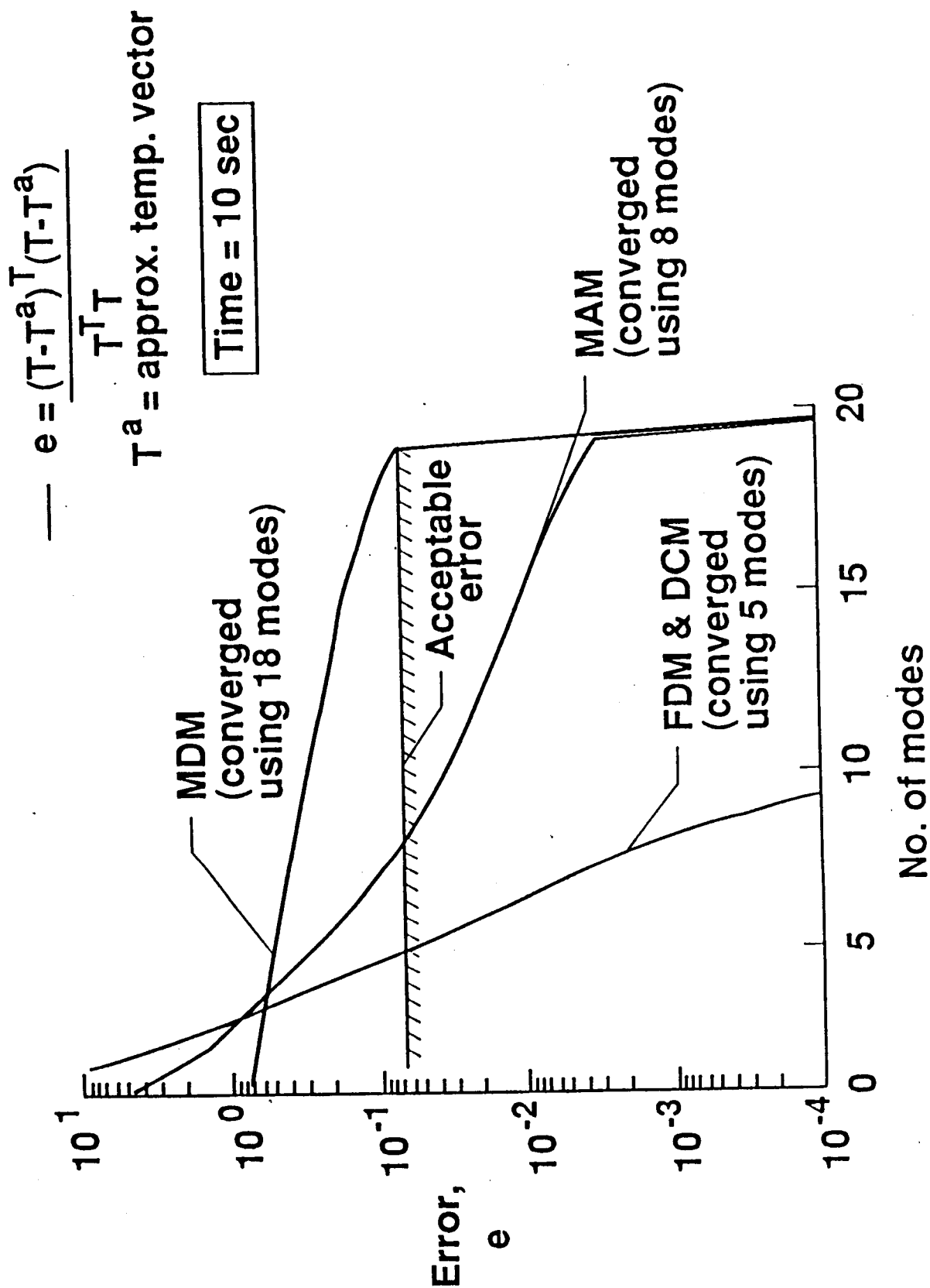


Figure 9.- Convergence of modal methods for a one-dimensional, transient thermal problem at $t = 10$ sec.



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16. Abstract The paper presents and evaluates a force-derivative method which produces higher-order modal solutions to transient problems. These higher-order solutions converge to an accurate response using fewer degrees-of-freedom (eigenmodes) than lower-order methods such as the mode-displacement or mode-acceleration methods. Results are presented for non-proportionally damped structural problems as well as thermal problems modeled by finite elements.					
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